PROBLEMS / SOLUTIONS / COMMENTS
Round 3 - Year 10 - Academic Year 1998-99

Gene A. Berg <gaberg@ieee.org>, Editor

1/3/10. Determine the leftmost three digits of the number
$$1^1 + 2^2 + 3^3 + \ldots + 999^{999} + 1000^{1000}.$$

Solution 1 by Jason Oh (11/MD): The leftmost three numbers of this sum are 100. To prove this I will show that the only term that contributes to the leftmost three digits is the 1000^{1000} term. The way to see this is by considering the series
$$1000^1 + 1000^2 + 1000^3 + \ldots + 1000^{999} + 1000^{1000}$$
Without much consideration, this series can be seen to add up to
$$100100100100\ldots 1000$$
In this series the terms from 1000^1 to 1000^{999} have no effect on the three leftmost digits. In the original series, the sum of the terms from 1^1 to 999^{999} will be smaller than the sum of the terms 1000^1 to 1000^{999} in the second series since each term is smaller and the original series is dominated by the second series. Thus in the original series, the first 999 terms will also not contribute to the leftmost three digits of the sum. The leftmost three terms of the sum come from the 1000^{1000} term, and are therefore 100.

Solution 2 by Marcus Aaron (10/TX): Setting
$$x = 1^1 + 2^2 + 3^3 + \ldots + 999^{999} + 1000^{1000}$$
it is clear that
$$1000^{1000} < x < 1000^1 + 1000^2 + 1000^3 + \ldots + 1000^{999} + 1000^{1000}$$
or that x is between 1000000... and 100100100... . Since both these numbers have exactly 3001 digits, the first three digits of x must be 100.

Editor's comment: Thanks are due to Professor Gregory Galperin of Eastern Illinois University for this nice problem.

2/3/10. There are infinitely many ordered pairs (m, n) of positive integers for which the sum
$$m + (m + 1) + (m + 2) + \ldots + (n - 1) + n$$
is equal to the product mn. The four pairs with the smallest values of m are *(1, 1), (3, 6), (15, 35)*, and *(85, 204)*. Find three more *(m, n)* pairs.

Solution 1 by Lucy Jiang (12/MD): The equation can be rewritten as:

$$(m+n) \times \frac{n-m+1}{2} = mn,$$

which simplifies to

$$m^2 - m(1-2n) - n^2 - n = 0.$$

Although 0 is not a positive integer, it can be used in determining a pattern for m and n because $0 + 0 = 0 \times 0$.

Suppose $(m_0, n_0) = (0, 0)$, $(m_1, n_1) = (1, 1)$, $(m_2, n_2) = (3, 6)$, and so on. We observe that $n_{i+2} = 6n_{i+1} - n_i$, as shown below:

$$(0, 0)$$
$$(1, 1) \qquad\qquad 1 \times 6 - 0 = 6$$
$$(3, 6) \qquad\qquad 6 \times 6 - 1 = 35$$
$$(15, 35) \qquad\qquad 35 \times 6 - 6 = 204$$
$$(85, 204)$$

Using this pattern, the next value for n is $204 \times 6 - 35 = 1189$. Substituting n into the equation $m^2 - m(1-2n) - n^2 - n = 0$, we obtain $m = 493$. Thus the ordered pair is **(493, 1189)**, and this works.

By the same method, two other ordered pairs are: **(2871, 6930)** and **(16731, 40391)**.

Solution 2 by Trevor Bass (12/NY): Using the fact that the sum $1 + 2 + 3 + \ldots + i = \frac{i(i+1)}{2}$, where i is an integer, we can rewrite the equation

$$m + (m+1) + (m+2) + \ldots + (n-1) + n = mn$$

as

$$\frac{n(n+1)}{2} - \frac{m(m+1)}{2} = mn,$$

so

$$m^2 + (2n-1)m + (-n^2 - n) = 0.$$

In order for m to be an integer, the discriminant of this quadratic equation must be the square of an integer, so

$$(2n-1)^2 - 4(-n^2 - n) = i^2,$$

where i is an arbitrary integer. This equation can be rewritten as

$$8n^2 + 1 = i^2.$$

The first seven values of n for which there exist i's are: 1, 6, 35, 204, 1189, 6930, and 40391. For each n, the corresponding m can be computed by solving the quadratic equation $m^2 + (2n-1)m + (-n^2 - n) = 0$. The first four n's are given in the problem statement, so the next three *(n, m)* pairs are **(493, 1189)**, **(2871, 6930)**, and **(16731, 40391)**.

Solution 3 by David Walker (11/NE): The expression $m + (m + 1) + \ldots + (n - 1) + n$ can be written $(m + n) \times \dfrac{n - m + 1}{2}$. Set this equal to mn and obtain $(m + n)(n - m + 1) = 2mn$. If you put 1, 3, 15, or 85 in for m in this equation, you get two solutions for each m. The solutions for 1 are 1 and 0. The solutions for 3 are 6 and -1. The solutions for 15 are 35 and -6, and for 85 are 204 and -35. I noticed the pattern that the solution to an m that is negative is the positive solution to the previous m value. So I put -204 into the equation and got 493. Then I put -493 back in for m and got 1189. I continued doing this and got the three (m, n) pairs **(493, 1189), (2871, 6930)**, and **(16731, 4039)**.

Solution 4 by Oaz Nir (10/CA): Three pairs are: $(m, n) =$ **(493, 1189), (2871, 6930)**, and **(16731, 40391)**.

Claim: If *(m, n)* is a solution, then *(2n + m, 5n + 2m - 1)* also is.

Using this claim, we easily recursively compute the three pairs above. Starting with the given solution (85, 204), we use the transformation $(m, n) \to (2n + m, 5n + 2m - 1)$ three times in succession.

To prove the claim, we first use the arithmetic series sum formula to transform the given equation into the following equivalent form:
$$(\textit{sum of the first and last terms})(\textit{number of terms})/2 = mn$$
$$(m + n)(n - m + 1) = 2mn.$$
Now we see that $(2n + m, 5n + 2m - 1)$ is a solution if
$$([2n + m] + [5n + 2m - 1])([5n + 2m - 1] - [2n + m] + 1) = 2[2n + m][5n + 2m - 1]$$
$$\Leftrightarrow (7n + 3m - 1)(3n + m) = 2[2n + m][5n + 2m - 1]$$
$$\Leftrightarrow 21n^2 + 16mn + 3m^2 - 3n - m = 20n^2 + 18mn - 4n + 4m^2 - 2m$$
$$\Leftrightarrow n^2 + n - m^2 + m = 2mn$$
$$\Leftrightarrow (n^2 + n - mn) + (-m^2 + mn + m) = 2mn$$
$$\Leftrightarrow n(n - m + 1) + m(n - m + 1) = 2mn$$
$$\Leftrightarrow (n + m)(n - m + 1) = 2mn.$$
That is, if *(m, n)* is a solution then *(2n + m, 5n + 2m - 1)* is also a solution.

Solution 5 by Reid Barton (10/MA): Answer: **(493, 1189), (2871, 6930)**, and **(16731, 40391)**.

Since $m + (m + 1) + \ldots + (n - 1) + n = (n - m + 1)(m + n)/2$, we want to find m and n such that
$$mn = \frac{(n - m + 1)(m + n)}{2}.$$
We have
$$mn = \frac{(n - m + 1)(m + n)}{2}.$$

$$\Leftrightarrow 2mn = nm + n^2 - m^2 - mn + m + n$$
$$\Leftrightarrow m^2 + 2mn - n^2 - m - n = 0$$
$$\Leftrightarrow (m+n)^2 - (m+n) - 2n^2 = 0$$
$$\Leftrightarrow 4(m+n)^2 - 4(m+n) + 1 - 8n^2 = 1$$
$$\Leftrightarrow (2m+2n-1)^2 - 8n^2 = 1$$
$$\Leftrightarrow t^2 - 8n^2 = 1$$

This is Pell's equation with primitive solution (3,1), so its solutions are given by

$t - n\sqrt{8} = (3 - \sqrt{8})^k$ for $k \geq 1$. For $k = 1, 2, 3$, and 4 we obtain $(t, n) = (3, 1), (17, 6), (99, 35)$, (577, 204), which give $(m, n) = (1, 1), (3, 6), (15, 35), (85, 204)$ respectively, as $m = (t-2n +1)/2$. Letting $k = 5, 6$, and 7 gives $(t, n) = (3363, 1189), (19601, 6930)$, and $(114243, 40391)$, which give $(m, n) = (493, 1189), (2871, 6930), (16731, 40391)$, three more solutions to the original equation.

Solution 6 by Mike Fliss (12/NJ): The sum on the left side of the equation (the sum of the integers from m to n) can be rewritten as $(n - m + 1)(n + m)/2$ since this is the number of terms times the average term. We have

$$(n - m + 1)(n + m)/2 = nm$$
$$n^2 - m^2 + n + m = 2nm$$

Now, in $Ax^2 + Bx + C = 0$ form, this is

$$n^2 + (1 - 2m)n + (m - m^2) = 0$$

We use the quadratic formula to obtain solutions for n. We want positive solutions for n. For all $m \geq 1$, observe that $2m - 1 \leq \sqrt{8m^2 - 8m + 1}$, so we do not consider the $\pm$, just the + in the quadratic formula to get positive values for n. So

$$n = \frac{2m - 1 + \sqrt{8m^2 - 8m + 1}}{2}$$

$$n = m + \left[\frac{\sqrt{8m^2 - 8m + 1} - 1}{2} \right]$$

Set

$$k = \frac{\sqrt{8m^2 - 8m + 1} - 1}{2}$$

Since m and n are integers, by the additive closure of the integers, k is also an integer.

$$\sqrt{8m^2 - 8m + 1} = 2k + 1$$
$$8m^2 - 8m + 1 = (2k + 1)^2$$
$$8m^2 - 8m + 2 = (2k + 1)^2 + 1$$

$$(2k + 1)^2 - 2(m - 1)^2 = -1$$
$$s^2 - 2r^2 = -1$$

where $s = 2k + 1$ and $r = m - 1$. This is Pell's equation, solvable by a continued fraction expansion of $\sqrt{2}$, because

$$\sqrt{2} = \sqrt{\frac{s^2 + 1}{r^2}}.$$

Continued fraction expansion yields fractions (s/r). If (s, r) is a solution to the equation $s^2 - 2r^2 = \pm 1$, then the following argument shows that $(s+2r, s+r)$ is also a solution:

$$(s + 2r)^2 - 2(s + r)^2 = s^2 + 4sr + 4r^2 - 2s^2 - 4sr - 2r^2$$
$$= -s^2 + 2r^2$$
$$= \mp 1$$

Therefore, the sequence of fractional expansions follows this form:

$$\frac{s_i}{r_i}, \quad \frac{s_i + 2r_i}{s_i + r_i}, \quad \dots$$

Thus, the answers for the (s/r) expansion are

1	3	7	17	41	99	239	577	1393	3363	8119	19601	47321	114243	275807
1	2	5	12	29	70	169	408	985	2378	5741	13860	33461	80782	195025

BUT, these are the answers for +1 and -1; we only want solutions for -1. Therefore, we take every other term.

1	7	41	239	1393	8119	47321	275807
1	5	29	169	985	5741	33461	195025

Since the bottom number corresponds to r, and $2m-1 = r$, we get the m values (first 7) in the table

below. Plugging these m's back into the original $n = m + \left[\dfrac{\sqrt{8m^2 - 8m + 1} - 1}{2}\right]$, we get the cor-

responding n's.

m	1	3	15	85	**493**	**2871**	**16731**
n	1	6	35	204	**1189**	**6930**	**40391**

Three new pairs (m, n) are listed in bold in the table.

Editor's comment: We are indebted to Professor Suresh T. Thaker of India for this excellent problem.

3/3/10. The integers from 1 to 9 can be arranged into a 3×3 array (as shown on the right) so that the sum of the numbers in every row, column, and diagonal is a multiple of 9.

A	B	C
D	E	F
G	H	I

(a.) Prove that the number in the center of the array must be a multiple of 3.
(b.) Give an example of such an array with 6 in the center.

Solution 1 by Rachel Johnson (9/MN): (a) There are nine possible combinations of three distinct numbers from 1 to 9 that have a sum of 9 or 18 (multiples of 9). They are: $\{1, 2, 6\}$ and $\{2, 3, 4\}$ have a sum of 9, while $\{1, 8, 9\}$, $\{2, 7, 9\}$, $\{3, 6, 9\}$, $\{3, 7, 8\}$, $\{4, 5, 9\}$, and $\{4, 6, 8\}$ have a sum of 18. These combinations have either one or three multiples of 3 in each of them.

Since all possible combinations have a multiple of 3 and there are only three multiples of 3, the multiples of 3 must be shared. This can be done by placing two (of 3, 6, and 9) in opposite corners. This provides a multiple of 3 for the four outside combinations. To provide a multiple of three for the other combinations, the remaining multiple of 3 would have to be placed in the center.

9	1	8
5	6	7
4	2	3

(b) At the right is a possible arrangement of the array with 6 in the middle.

Solution 2 by Michelle Rengarajan (8/CA): (a) According to the given information:
1) $A + B + C = 9k_1$
2) $D + E + F = 9k_2$
3) $G + H + I = 9k_3$
4) $A + D + G = 9k_4$
5) $B + E + H = 9k_5$
6) $C + F + I = 9k_6$
7) $A + E + I = 9k_7$
8) $C + E + G = 9k_8$

1	8	9
5	6	7
3	4	2

Solve for E in equations 2, 7, and 8.

$E = 9k_2 - (D + F)$
$E = 9k_7 - (A + I)$
$E = 9k_8 - (C + G)$

Add these three equations together to find the value of 3E:

$3E = 9k_2 + 9k_7 + 9k_8 - (A + D + G + C + F + I)$.

So, according to equations 4 and 6

$3E = 9k_2 + 9k_7 + 9k_8 - 9k_4 - 9k_6$.

Therefore, 3E is a multiple of 9, so E must be a multiple of 3.

7	8	3
2	6	1
9	4	5

(b) An array with 6 in the middle is shown on the right.

Editor's comment: This clever problem was posed by Dr. Erin Schram of the National Security Agency. He claims there are 24 possible arrays for part (b), all of which are rotations or reflections of one of the three arrays shown above.

4/3/10. Prove that if $0 < x < \pi/2$, then

$$\sec^6 x + \csc^6 x + (\sec^6 x)(\csc^6 x) \geq 80.$$

Solution 1 by Lucy Jin (11/MI): (Proof by contradiction.)

$$\sec^6 x + \sec^6 x + (\sec^6 x)(\csc^6 x)$$

$$= \frac{1}{\cos^6 x} + \frac{1}{\sin^6 x} + \frac{1}{(\cos^6 x)(\sin^6 x)}$$

$$= \frac{1}{(\cos^2 x)^3} + \frac{1}{(\sin^2 x)^3} + \frac{1}{(\cos^6 x)(\sin^6 x)}$$

$$= \frac{(\sin^2 x)^3 + (\cos^2 x)^3 + 1}{(\sin^6 x)(\cos^6 x)}$$

$$= \frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x) + 1}{(\sin^6 x)(\cos^6 x)}$$

$$= \frac{\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x + 1}{(\sin^6 x)(\cos^6 x)}$$

$$= \frac{(\sin^2 x + \cos^2 x)^2 - 3\sin^2 x \cos^2 x + 1}{(\sin^6 x)(\cos^6 x)}$$

$$= \frac{2 - 3\sin^2 x \cos^2 x}{(\sin^6 x)(\cos^6 x)}$$

The last expression is equivalent to the original trigonometric expression.

Now we begin the proof by contradiction. Assume

$$\frac{2 - 3\sin^2 x \cos^2 x}{(\sin^6 x)(\cos^6 x)} < 80.$$

We show that this assumption leads to a contradiction, and therefore must be false.

Let $u = \sin x \cos x$. The inequality above can be written

$$\frac{2 - 3u^2}{u^6} < 80$$

$$80u^6 + 3u^2 - 2 > 0$$

$$\left(u + \frac{1}{2}\right)\left(u - \frac{1}{2}\right)(80u^4 + 20u^2 + 8) > 0$$

Observe $80u^4 + 20u^2 + 8$ is always positive. Therefore $\left(u + \frac{1}{2}\right)\left(u - \frac{1}{2}\right) > 0$, so $u^2 > \frac{1}{4}$ and either

$$u < -\frac{1}{2} \qquad\qquad \textbf{or} \qquad\qquad u > \frac{1}{2}. \qquad \text{Recall } u = \sin x \cos x, \text{ so}$$

$$\sin x \cos x < -\frac{1}{2} \qquad\qquad\qquad\qquad \sin x \cos x > \frac{1}{2}$$

$$\sin 2x < -1 \qquad\qquad\qquad\qquad\qquad \sin 2x > 1.$$

Both these cases are contradictory, because $-1 \le \sin\varphi \le 1$.

Therefore, our assumption was false and we must have

$$\frac{2 - 3(\sin^2 x \cos^2 x)}{(\sin^6 x)(\cos^6 x)} \ge 80.$$

and $\sec^6 x + \csc^6 x + (\sec^6 x)(\csc^6 x) \ge 80$.

**Solution 2 by Mark Tong (12/VA):** This proof makes extensive use of the Arithmetic Mean-Geometric Mean (AM-GM) Inequality, which states that for positive numbers a and b, $(a + b)/2 \ge (ab)^{1/2}$. For the first step of the proof, substitute into AM-GM with $a = \sin^2 x$ and $b = \cos^2 x$. So $(\sin^2 x \cos^2 x)^{1/2} \le (\sin^2 x + \cos^2 x)/2 \qquad \Rightarrow \qquad \sin x \cos x \le 1/2$

$\Rightarrow \quad \sin^6 x \cos^6 x \le 1/64 \qquad \Rightarrow \qquad \sec^6 x \csc^6 x \ge 64.$

Using this new equation and substituting into AM-GM again with $a = \sec^6 x$ and $b = \csc^6 x$:

$(\sec^6 x + \csc^6 x)/2 \ge (\sec^6 x \csc^6 x)^{1/2} \qquad \Rightarrow \qquad \sec^6 x + \csc^6 x \ge 2(64)^{1/2} = 16.$

Since $\sec^6 x \csc^6 x \ge 64$ and $\sec^6 x + \csc^6 x \ge 16$,

$$\sec^6 x + \csc^6 x + \sec^6 x \csc^6 x \ge 80.$$

5/3/10. In the figure on the right, O is the center of the circle, OK and OA are perpendicular to one another, M is the midpoint of OK, BN is parallel to OK, and $\angle AMN = \angle NMO$. Determine the measure of $\angle ABN$ in degrees.
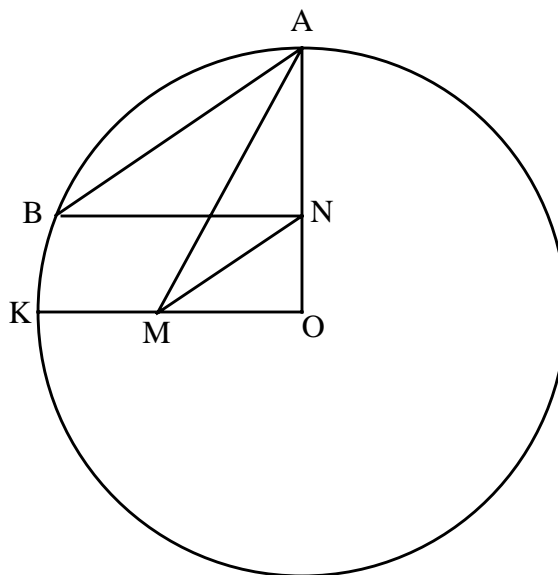
Solution by Chi-Bong Chan (12/NJ): For convenience and without loss of generality let the radius of the circle be 2. Then $OM = 1$, $OA = 2$, and the Pythagorean Theorem gives $MA = \sqrt{5}$. The angle bisector theorem gives $MO/MA = NO/NA$. Rewrite as

$$\frac{MO}{MA + MO} = \frac{NO}{NA + NO}.$$

Since $AN + NO = OA = OB$, we have

$$\frac{NO}{OB} = \frac{MO}{MA + MO} = \frac{1}{\sqrt{5} + 1} = \frac{\sqrt{5} - 1}{4}.$$

If we reflect O over BN to get O', then $\dfrac{OO'}{OB} = \dfrac{2NO}{OB} = \dfrac{\sqrt{5} - 1}{2}.$

This is the golden mean, and one interesting fact involving the golden mean is that *the ratio of the side to the diagonal of a regular pentagon is the golden mean.*

Proof of the fact: Keep in mind that

1) because it is a regular pentagon, the five arcs are equal, and
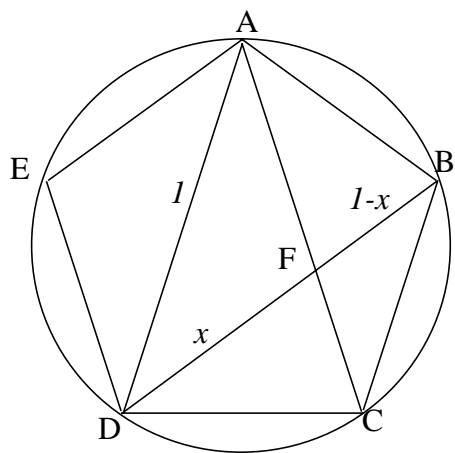
2) an angle subtending an arc has one-half the measure of the arc.

We shall first show that each diagonal is parallel to the opposite side, for example in the diagram of the regular pentagon below, $AD \parallel BC$. We have

$$\angle ADC = (arcAC)/2 = ((2/5)(360°))/2 = 72°,$$
$$\text{and } \angle BCD = (arcBAD)/2 = ((3/5)(360°))/2 = 108°.$$

Now $AD \parallel BC$ follows directly from the fact that $\angle ADC$ and $\angle BCD$ are complementary.
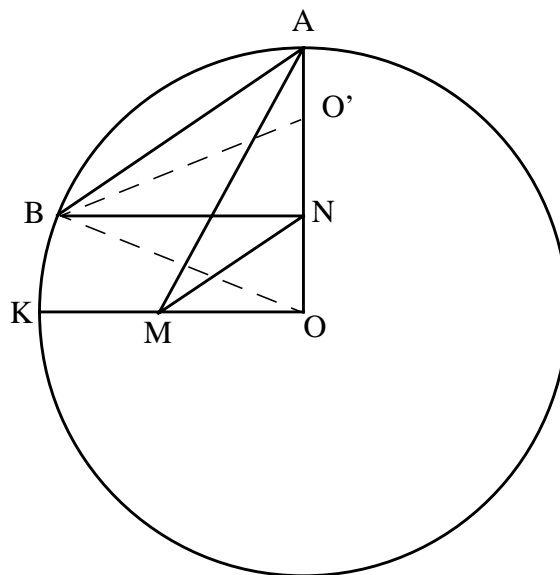
Now, without loss of generality, let x be the length of a side and 1 be the length of a diagonal, so that their ratio is simply x. Since each diagonal is parallel to the opposite side, $AFDE$ is a parallelogram, and therefore $DF = AB = x$. Moreover, since $AD \parallel BC$, $\angle CBF = \angle FDA$ and $\angle BCF = \angle FAD$. Therefore, $\triangle ADF = \triangle BCF$, so $(BC)/(DA) = (BF)/(DF)$. That means

$x/1 = (1-x)/x$ or $x^2 = 1-x$, which has the *golden mean* as the positive root, as desired. This completes the proof of the fact.

Back to our problem, we see that OBO' would be similar to DAC in the pentagon because their sides are proportional (OO'/BO and DC/DA are both equal to the golden mean, and both triangles are isosceles). Therefore $\angle BOO' = \angle ADC$ of the pentagon, and so $\angle BOO' = 72°$ from the earlier calculation.

Finally, to find $\angle ABN$, reflect B across AO to get B'. Then $\angle B' = (arcAB)/2 = (72°)/2 = 36°$ because an angle subtending an arc has half the measure of the arc. Since $\angle B'$ is the image of $\angle ABN$ under the reflection, we have at last $\angle ABN = \angle B' = \mathbf{36°}$.
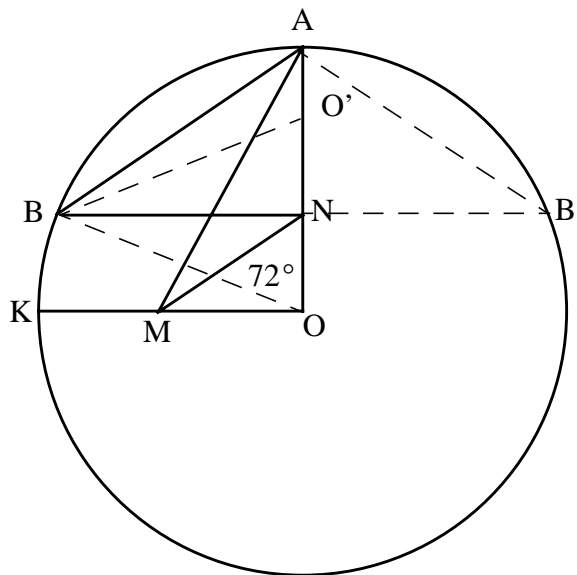
Editor's comment: Again we thank Professor Gregory Galperin of Eastern Illinois University for this, his second beautiful problem included in this round.